Asymptotic models for the multiple electromagnetic wave scattering problem by small obstacles

Justine Labat, Victor Péron, Sébastien Tordeux

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Journées Ondes Sud-Ouest

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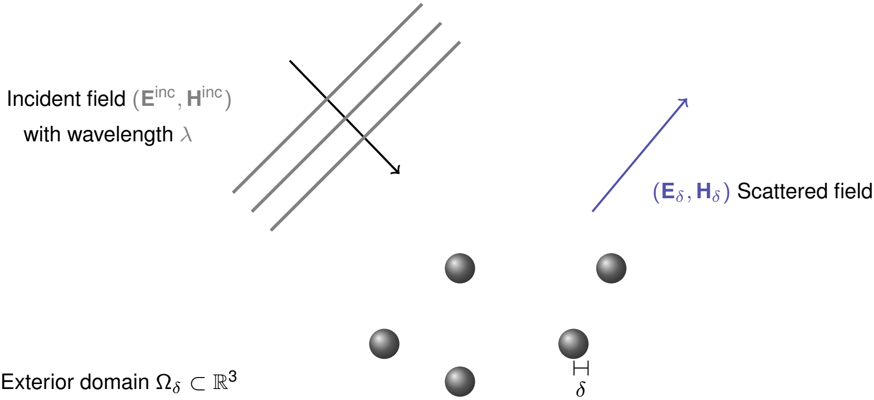
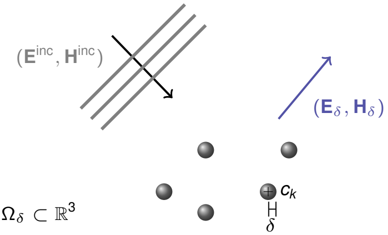
3D Scattering problem by small obstacles



Asymptotic assumption : $\delta \ll \lambda$

Model problem



- Time-harmonic domain
- Homogeneous & isotropic medium Ω_δ
- Perfect conductors of characteristic length δ

Time-harmonic Maxwell equations

$$\begin{cases} \mathbf{curl}\,\mathbf{E}_\delta - i\kappa\,\mathbf{H}_\delta = 0 & \text{in } \Omega_\delta \\ \mathbf{curl}\,\mathbf{H}_\delta + i\kappa\,\mathbf{E}_\delta = 0 & \text{in } \Omega_\delta \end{cases}$$

with $\kappa^2 = \omega^2\mu(\varepsilon + \frac{i\sigma}{\omega})$, $\Im(\kappa) \geq 0$

Boundary condition

$$\mathbf{n} \times \mathbf{E}_\delta = -\mathbf{n} \times \mathbf{E}^{inc} \qquad \text{on } \partial\Omega_\delta$$

Silver-Müller radiation condition

$$r\big(\mathbf{H}_\delta \times \widehat{x} - \mathbf{E}_\delta\big) \xrightarrow[r\to\infty]{} 0 \quad \text{unif. in } \widehat{x} = \frac{\mathbf{x}}{r}$$

Mathematical well-posedness

For any $\mathbf{E}^{inc} \in \mathbf{H}_{loc}(\mathbf{curl}\,, \Omega_\delta)$ there exist a unique solution $(\mathbf{E}_\delta, \mathbf{H}_\delta) \in \mathbf{H}_{loc}(\mathbf{curl}\,, \Omega_\delta)^2$ to the exterior Maxwell problem.

Applications

- Medical imaging
- Civil engineering
- Nuclear industry
- Atmospheric science
- Astronomy
- Climatology
- ...

Inverse problem ?

- Locate heterogeneities, cracks, ...
- Characterize them

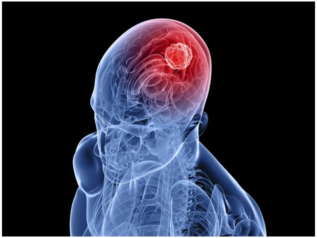


Figure: Small tumor detection[1]

Figure: Non-destructive testing[2]

[1] https://medcitynews.com/wp-content/uploads/2017/01/GettyImages-94456546-600x450.jpg
[2] https://tri-intl.com/wp-content/uploads/2017/07/NonDestructiveTesting.jpg

Workflow

Asymptotic models

- ✗ Restricted to small obstacles
- ✓ Low computational cost
- ✓ Meshless method

Workflow

Multiple scattering →

Asymptotic models

- ✗ Restricted to small obstacles
- ✓ Low computational cost
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Foldy-Lax model

- ✓ Interactions taken into account
- ✓ Low computational cost
- ✓ Meshless method

↓ Superposition principle

Born approximation

- ✗ No interaction between the obstacles
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(Non-exhaustive) alternatives

- Boundary integral equations
  - Boundary element methods
  - Spectral-based methods
- Enrichment of approximation spaces
  - Treffftz-based methods
  - Extended FEM
  - Partition of Unity method

Workflow

Multiple scattering →

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Some references

- Historic references
  - Rayleigh (1884), Foldy (1945), Lax (1951)
- Small defect theory
  - Il'In (1992), Maz'ya-Nazarov-Plamenevskii (2000)

- Acoustic obstacle
  - Ammari-Kang (2003), Ramm (2005), Claeys (2008)
- Time-dependent domain
  - Mattesi (2014), Korikov (2015), Marmorat (2015)
- Electromagnetic obstacle
  - Vogelius-Volkov (2000), Ammari-Vogelius-Volkov (2001), Korikov-Plamenevskii (2017)

- Foldy theory
  - Martin (2004), Cassier-Hazard (2013), Bendali-Cocquet-Tordeux (2014), Challa-Hu-Sini (2014)
- High-order spectral algorithms
  - Xu (1995), Ganesh-Hawkins (2009), Barucq-Chabassier-Pham-Tordeux (2017)
- Inverse problem
  - Volkov (2001), Ammari-Kang (2004), Challa-Sini (2012)

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Outline

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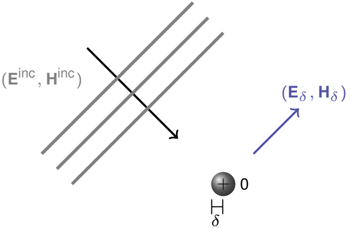
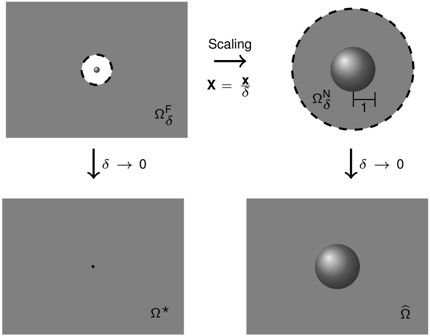
Approximation of solution to single scattering

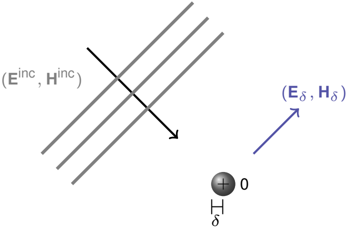
Method of matched asymptotic expansions

- Domain decomposition
- Local approximations
- Matching procedure





Approximation of solution to single scattering



Method of matched asymptotic expansions

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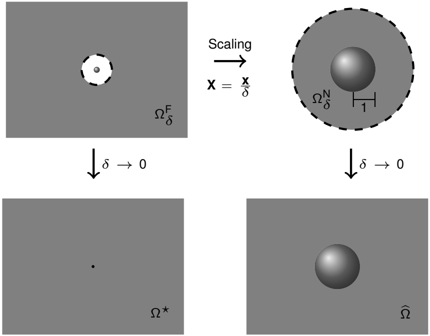

Asymptotic expansions

- **Far field** expansion in $\Omega^\star = \mathbb{R}^3 \setminus \{0\}$
- **Near field** expansion in $\widehat{\Omega} = \mathbb{R}^3 \setminus \overline{\mathcal{B}(0,1)}$

Approximation of solution to single scattering

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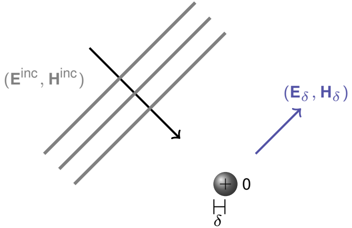

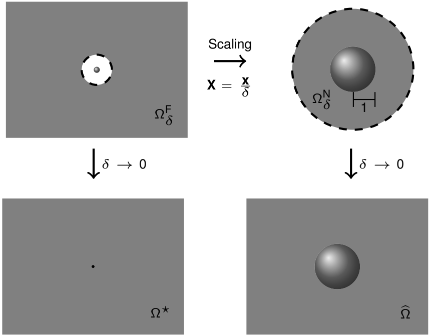

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- **Far field** expansion in $\Omega^\star = \mathbb{R}^3 \setminus \{0\}$
- **Near field** expansion in $\widehat{\Omega} = \mathbb{R}^3 \setminus \overline{\mathcal{B}(0,1)}$

related by an asymptotic process

Asymptotic expansions

Near field expansion

$$\mathbf{E}_\delta(\delta\mathbf{X}) \approx \widehat{\mathbf{E}}_0(\mathbf{X}) + \delta\widehat{\mathbf{E}}_1(\mathbf{X}) + \delta^2\widehat{\mathbf{E}}_2(\mathbf{X}) + \dots$$

$$\mathbf{H}_\delta(\delta\mathbf{X}) \approx \widehat{\mathbf{H}}_0(\mathbf{X}) + \delta\widehat{\mathbf{H}}_1(\mathbf{X}) + \delta^2\widehat{\mathbf{H}}_2(\mathbf{X}) + \dots$$

Far field expansion

$$\mathbf{E}_\delta(\mathbf{x}) \approx \delta^3\widetilde{\mathbf{E}}_3(\mathbf{x}) + \delta^4\widetilde{\mathbf{E}}_4(\mathbf{x}) + \delta^5\widetilde{\mathbf{E}}_5(\mathbf{x}) + \dots$$

$$\mathbf{H}_\delta(\mathbf{x}) \approx \delta^3\widetilde{\mathbf{H}}_3(\mathbf{x}) + \delta^4\widetilde{\mathbf{H}}_4(\mathbf{x}) + \delta^5\widetilde{\mathbf{H}}_5(\mathbf{x}) + \dots$$

$\mathbf{X} = \frac{\mathbf{x}}{\delta}$: fast variable

- For an obstacle of arbitrary shape

Numerical solution of elementary problems independent of δ
  - Near-field: quasi-static problems
  - Far-field: time-harmonic problems + equivalent multipole distributions

- For a spherical obstacle

  - Analytical solutions of elementary problems
  - Equivalent multipole distributions

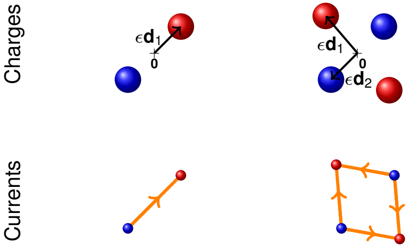
Near-field approximation: Spherical case

Close to the obstacle : quasi-static approximation

Superposition of electric fields generated by multipole distributions:

$$\widehat{\mathbf{E}}_p = \sum_{\substack{n=1 \\ |n-p|\ \mathrm{odd}}}^{p+1} \mathcal{E}^{\mathrm{stat}}_{n,\mathrm{elec}}[\mathbf{u}^{p,\mathrm{E}}_{1,n}, \ldots, \mathbf{u}^{p,\mathrm{E}}_{n,n}] + \sum_{\substack{n=1 \\ |n-p|\ \mathrm{even}}}^{p+1} \mathcal{E}^{\mathrm{stat}}_{n,\mathrm{mag}}[\mathbf{u}^{p,\mathrm{H}}_{1,n}, \ldots, \mathbf{u}^{p,\mathrm{H}}_{n,n}] \qquad p = 0, 1, 2, \ldots$$

Dipole ($n = 1$) Quadrupole ($n = 2$)



$\mathcal{E}^{\mathrm{stat}}_n[\mathbf{d}_1, \ldots, \mathbf{d}_n]$ electric fields

- generated by charges OR currents
- electric/magnetic 2^n-point charges
  - electric charge: $\boldsymbol{\mathcal{E}} = -\nabla\mathcal{V}$
  - magnetic current: $\boldsymbol{\mathcal{E}} = -\mathbf{curl}\,\boldsymbol{\mathcal{A}}$
- of moments $(\mathbf{d}_1, \ldots, \mathbf{d}_n)$
- defined by an asymptotic process

Near-field approximation: Spherical case

Close to the obstacle : quasi-static approximation

Superposition of electric fields generated by multipole distributions:

$$\widehat{\mathbf{E}}_p = \sum_{\substack{n=1 \\ |n-p|\ \text{odd}}}^{p+1} \mathcal{E}^{\text{stat}}_{n,\text{elec}}[\mathbf{u}^{p,\text{E}}_{1,n}, \dots, \mathbf{u}^{p,\text{E}}_{n,n}] + \sum_{\substack{n=1 \\ |n-p|\ \text{even}}}^{p+1} \mathcal{E}^{\text{stat}}_{n,\text{mag}}[\mathbf{u}^{p,\text{H}}_{1,n}, \dots, \mathbf{u}^{p,\text{H}}_{n,n}] \qquad p = 0, 1, 2, \dots$$

	Order	Dipole		Quadrupole		Octupole	
		Electric	Magnetic	Electric	Magnetic	Electric	Magnetic
Near field	0	$\color{teal}{\bullet}$	$\color{red}{\bullet}$				
	1	$\color{red}{\bullet}$	$\color{teal}{\bullet}$	$\color{teal}{\bullet}$	$\color{red}{\bullet}$		
	2	$\color{teal}{\bullet}$	$\color{red}{\bullet}$	$\color{red}{\bullet}$	$\color{teal}{\bullet}$	$\color{teal}{\bullet}$	$\color{red}{\bullet}$

- $\color{teal}{\bullet}$ Electric field
- $\color{red}{\bullet}$ Magnetic field

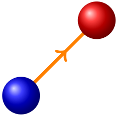
Far-field approximation

Far from the obstacle: time-harmonic approximation

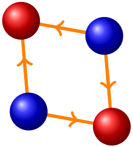
Superposition of electric fields generated by multipole distributions:

$$\widetilde{\mathbf{E}}_{p+3} = \sum_{n=1}^{p+1} \mathcal{E}_{n,\mathrm{elec}}[\mathbf{v}^{p,\mathrm{E}}_{1,n}, \dots, \mathbf{v}^{p,\mathrm{E}}_{n,n}] + \mathcal{E}_{n,\mathrm{mag}}[\mathbf{v}^{p,\mathrm{H}}_{1,n}, \dots, \mathbf{v}^{p,\mathrm{H}}_{n,n}] \qquad p = 0, 1, 2, \dots$$

Dipole ($n = 1$)



Quadrupole ($n = 2$)



$\operatorname{div} \mathcal{J} - i\omega\varrho = 0$

$\mathcal{E}_n[\mathbf{d}_1, \dots, \mathbf{d}_n]$ electric fields

- generated by charges AND currents
  - related by charge conservation principle
- electric/magnetic 2^n-point charges
  - electric multipole: $\boldsymbol{\mathcal{E}} = -\nabla \mathcal{V}_{\mathrm{E}} + i\kappa \boldsymbol{\mathcal{A}}_{\mathrm{E}}$
  - magnetic multipole: $\boldsymbol{\mathcal{E}} = -\mathbf{curl}\, \boldsymbol{\mathcal{A}}_{\mathrm{H}}$
- of moments $(\mathbf{d}_1, \dots, \mathbf{d}_n)$
- defined by an asymptotic process

Far-field approximation

Far from the obstacle: time-harmonic approximation

$$\widetilde{\mathbf{E}}_{p+3} = \sum_{n=1}^{p+1} \mathcal{E}_{n,\mathrm{elec}}[\mathbf{v}_{1,n}^{p,\mathrm{E}}, \dots, \mathbf{v}_{n,n}^{p,\mathrm{E}}] + \mathcal{E}_{n,\mathrm{mag}}[\mathbf{v}_{1,n}^{p,\mathrm{H}}, \dots, \mathbf{v}_{n,n}^{p,\mathrm{H}}]$$

$$\widetilde{\mathbf{H}}_{p+3} = \sum_{n=1}^{p+1} \mathcal{H}_{n,\mathrm{mag}}[\mathbf{v}_{1,n}^{p,\mathrm{H}}, \dots, \mathbf{v}_{n,n}^{p,\mathrm{H}}] + \mathcal{H}_{n,\mathrm{elec}}[\mathbf{v}_{1,n}^{p,\mathrm{E}}, \dots, \mathbf{v}_{n,n}^{p,\mathrm{E}}]$$

	Order	Dipole		Quadrupole		Octupole	
		Electric	Magnetic	Electric	Magnetic	Electric	Magnetic
Far field	3	● ●	● ●				
	4	● ●	● ●	● ●	● ●		
	5	● ●	● ●	● ●	● ●	● ●	● ●

- Electric field
- Magnetic field

Far-field approximation

Far from the obstacle: time-harmonic approximation

$$\widetilde{\mathbf{E}}_{p+3} = \sum_{n=1}^{p+1} \mathcal{E}_{n,\text{elec}}[\mathbf{v}^{p,\text{E}}_{1,n}, \dots, \mathbf{v}^{p,\text{E}}_{n,n}] + \mathcal{E}_{n,\text{mag}}[\mathbf{v}^{p,\text{H}}_{1,n}, \dots, \mathbf{v}^{p,\text{H}}_{n,n}]$$

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- Spherical case

	Order	Dipole		Quadrupole		Octupole	
		Electric	Magnetic	Electric	Magnetic	Electric	Magnetic
Far field	3	• •	• •				
	4						
	5	• •	• •	• •	• •	• •	• •

- Collected dipolar approximation

	Order	Dipole		Quadrupole		Octupole	
		Electric	Magnetic	Electric	Magnetic	Electric	Magnetic
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	5	• •	• •				

Born approximation

- The electromagnetic fields are approximated by the superposition principle

$$\mathbf{E}_\delta(\mathbf{x}) \approx \sum_{k=1}^{N_{\text{obs}}} \mathbf{E}_{\delta,k}(\mathbf{x}) \qquad \mathbf{H}_\delta(\mathbf{x}) \approx \sum_{k=1}^{N_{\text{obs}}} \mathbf{H}_{\delta,k}(\mathbf{x})$$

- Each obstacle is modeled as a dipolar source around c_k

$$\mathbf{E}_{\delta,k}(\mathbf{x}) = \mathcal{E}_{1,\text{elec}}[\mathbf{d}^{\text{E}}_{\delta,k}](\mathbf{x} - c_k) + \mathcal{E}_{1,\text{mag}}[\mathbf{d}^{\text{H}}_{\delta,k}](\mathbf{x} - c_k)$$

For perfectly conducting spheres:

- Approximation of order 3

$$\mathbf{d}^{\text{E}}_{\delta,k} = 4\pi\delta^3 \mathbf{E}^{\text{inc}}(c_k) \qquad \mathbf{d}^{\text{H}}_{\delta,k} = -2\pi\delta^3 \mathbf{H}^{\text{inc}}(c_k)$$

- Collected dipolar approximation

$$\mathbf{d}^{\text{E}}_{\delta,k} = 4\pi\delta^3 \left(1 + \frac{3(\kappa\delta)^2}{10}\right) \mathbf{E}^{\text{inc}}(c_k) \qquad \mathbf{d}^{\text{H}}_{\delta,k} = -2\pi\delta^3 \left(1 - \frac{3(\kappa\delta)^2}{5}\right) \mathbf{H}^{\text{inc}}(c_k)$$

Foldy-Lax model

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- Vectorial formulation (3-order approximation)

  Find $\mathbf{d} = \left((\mathbf{d}^{\text{E}}_1), \dots, (\mathbf{d}^{\text{E}}_{N_{\text{obs}}}), (\mathbf{d}^{\text{H}}_1) \dots, (\mathbf{d}^{\text{H}}_{N_{\text{obs}}})\right)^\top \in \mathbb{C}^{6N_{\text{obs}}}$ such that

$$\mathbf{d} - \delta^3 \mathbb{A}\mathbf{d} = \delta^3 \mathbf{f}$$

with $\mathbf{f} = \begin{pmatrix} 4\pi\mathbf{E}^{\text{inc}}(c_1) \\ \vdots \\ 4\pi\mathbf{E}^{\text{inc}}(c_{N_{\text{obs}}}) \\ -2\pi\mathbf{H}^{\text{inc}}(c_1) \\ \vdots \\ -2\pi\mathbf{H}^{\text{inc}}(c_{N_{\text{obs}}}) \end{pmatrix}$ and $\mathbb{A}$ the "interaction" matrix

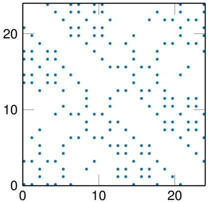

Figure: Skeleton of $\mathbb{A}$ for $N_{\text{obs}} = 4$

Foldy-Lax approximations

- Approximation of order 3

$$\mathbf{d} - \delta^3 \mathbb{A}\mathbf{d} = \delta^3 \mathbf{f}$$

- Collected dipolar approximation

$$\mathbb{D}^{-1}\mathbf{d} - \delta^3 \mathbb{A}\mathbf{d} = \delta^3 \mathbf{f}$$

where

$$\mathbb{D} = \mathrm{diag}(\underbrace{\alpha, \dots, \alpha}_{3N_{\mathrm{obs}}}, \underbrace{\beta, \dots, \beta}_{3N_{\mathrm{obs}}}) \qquad \alpha = 1 + \frac{3(\kappa\delta)^2}{10} \qquad \beta = 1 - \frac{6(\kappa\delta)^2}{10}$$

- Modified dipolar approximation

$$\widetilde{\mathbb{D}}^{-1}\mathbf{d} - \delta^3 \mathbb{A}\mathbf{d} = \delta^3 \mathbf{f}$$

where

$$\widetilde{\mathbb{D}} = \mathrm{diag}(\underbrace{\widetilde{\alpha}, \dots, \widetilde{\alpha}}_{3N_{\mathrm{obs}}}, \underbrace{\widetilde{\beta}, \dots, \widetilde{\beta}}_{3N_{\mathrm{obs}}}) \qquad \widetilde{\alpha} = \frac{3i}{2(\kappa\delta)^3} \frac{j_1(\kappa\delta)}{h_1^{(1)}(\kappa\delta)} \qquad \widetilde{\beta} = -\frac{3i}{(\kappa\delta)^3} \frac{j_1(\kappa\delta) + \kappa\delta j_1'(\kappa\delta)}{h_1^{(1)}(\kappa\delta) + \kappa\delta h_1^{(1)\prime}(\kappa\delta)}$$

Foldy-Lax approximations

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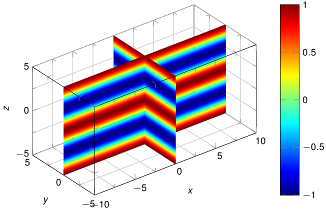
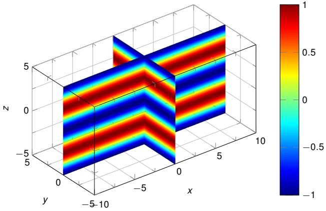
Numerical validation: Spherical case

- Incident field: Electromagnetic plane wave of wavelength $\lambda = 5$ along z-axis polarized by $\boldsymbol{e}_x$

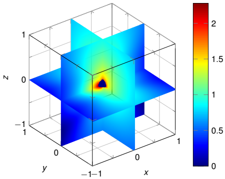


(a) Electric field (x-component)

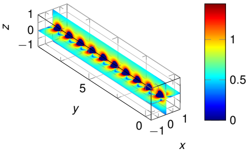


(b) Magnetic field (y-component)

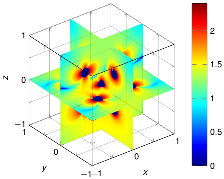
- Reference solution: "Spectral" solution truncated at the order 10



(c) One obstacle



(d) Aligned obstacles



(e) Real 3D-case

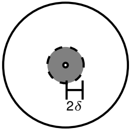
Single-scattering: Validation of asymptotic expansions

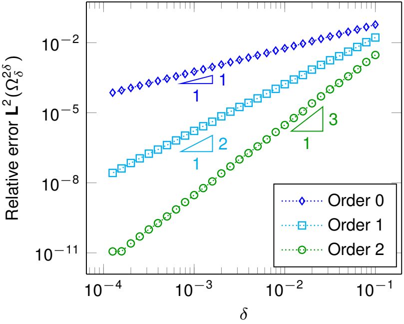
- Near-field approximations

Order 0 : $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$

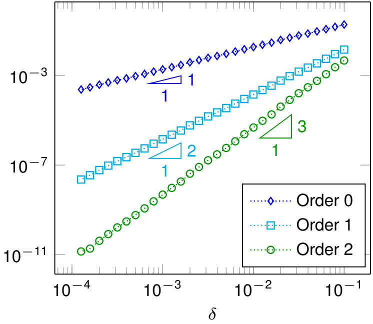
Order 1 : $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta}) - \delta\widehat{\mathbf{E}}_1(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$

Order 2 : $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta}) - \delta\widehat{\mathbf{E}}_1(\frac{\cdot}{\delta}) - \delta^2\widehat{\mathbf{E}}_2(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$





(f) Electric field



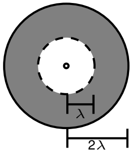
(g) Magnetic Field

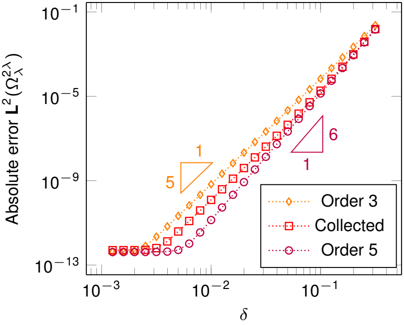
Single-scattering: Validation of asymptotic expansions

- Far-field approximations

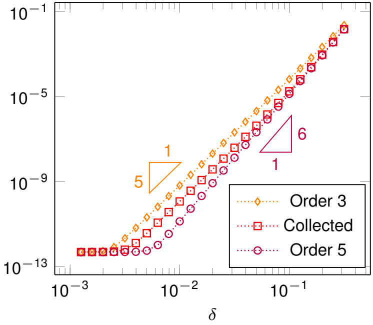
Order 3 : $\|\mathbf{E}_\delta - \delta^3\widetilde{\mathbf{E}}_3\|_{\mathbf{L}^2(\Omega_\lambda^{2\lambda})}$

Order 5 : $\|\mathbf{E}_\delta - \delta^3\widetilde{\mathbf{E}}_3 - \delta^5\widetilde{\mathbf{E}}_5\|_{\mathbf{L}^2(\Omega_\lambda^{2\lambda})}$





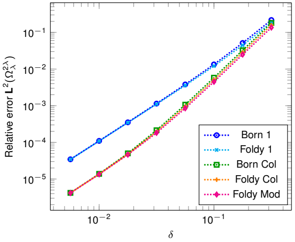
(l) Electric field



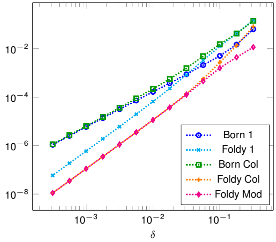
(m) Magnetic field

Multiple-scattering: Validation of Foldy-Lax model

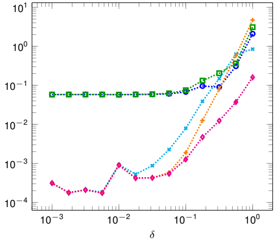
- Order 3: Born and Foldy
- Collected dipole improvement: Collected Born and Collected Foldy
- Spectral improvement: Modified Foldy



(n) Real 3D case with $N_{\text{obs}} = 13$ (fixed distance)



(o) Aligned obstacles with $N_{\text{obs}} = 5$ ($\sqrt{\delta}$-dependence)



(p) Aligned obstacles with $N_{\text{obs}} = 5$ (δ-dependence)

Outline

1. Asymptotic models
   - Single-scattering
   - Application: Born approximation
   - Multiple-scattering: Foldy-Lax model
   - Numerical results

2. Spectral method: Spherical case
   - Discretization
   - Numerical convergence
   - Comparison with asymptotic models
   - Comparison with finite element solutions

3. Conclusions and perspectives

Spectral method

- The electromagnetic fields are represented by the Stratton-Chu formula for $\mathbf{x} \in \Omega_\delta$

$$\mathbf{E}_\delta(\mathbf{x}) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{curl} \int_{\Gamma_\delta^k} \Phi(\mathbf{x}, \mathbf{y}) \mathbf{p}_k(\mathbf{y}) \, ds_{\mathbf{y}}$$

where $\Phi(\mathbf{x}, \mathbf{y}) = \frac{\exp(i\kappa|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}$

Spectral method

- The electromagnetic fields are represented by the Stratton-Chu formula for $\mathbf{x} \in \Omega_\delta$

$$\mathbf{E}_\delta(\mathbf{x}) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{curl} \int_{\Gamma_\delta^k} \Phi(\mathbf{x},\mathbf{y})\mathbf{p}_k(\mathbf{y})\, ds_{\mathbf{y}}$$

- Each $\mathbf{p}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$ satisfies the boundary integral equation

$$\sum_{j=1}^{N_{\text{obs}}} \mathcal{M}_\Gamma^{kj}\mathbf{p}_j = -\mathbf{n} \times \mathbf{E}^{\text{inc}} \qquad \text{on } \Gamma_\delta^k$$

where $\mathcal{M}_\Gamma^{kj} : \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_\delta^j}, \Gamma_\delta^j) \longrightarrow \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$ is the extension of

$$\mathcal{M}_\Gamma^{kj}\boldsymbol{\lambda}(\mathbf{x}_\Gamma) = \mathbf{n}(\mathbf{x}_\Gamma) \times \lim_{\mathbf{x}\to\mathbf{x}_\Gamma} \mathbf{curl} \int_{\Gamma_\delta^j} \Phi(\mathbf{x},\mathbf{y})\boldsymbol{\lambda}(\mathbf{y})\, ds_{\mathbf{y}} \qquad \boldsymbol{\lambda} \in \mathscr{C}^\infty(\Gamma_\delta^j) \qquad \mathbf{x}_\Gamma \in \Gamma_\delta^k$$

Spectral method

- The electromagnetic fields are represented by the Stratton-Chu formula for $\mathbf{x} \in \Omega_\delta$

$$\mathbf{E}_\delta(\mathbf{x}) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{curl} \int_{\Gamma_\delta^k} \Phi(\mathbf{x}, \mathbf{y}) \mathbf{p}_k(\mathbf{y}) \, ds_{\mathbf{y}}$$

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$$\sum_{j=1}^{N_{\text{obs}}} \mathcal{M}_\Gamma^{kj} \mathbf{p}_j = -\mathbf{n} \times \mathbf{E}^{\text{inc}} \qquad \text{on } \Gamma_\delta^k$$

- Galerkin discretization of the BIE on local spectral basis with N_{mod} modes

$$\mathbf{p}_j(\mathbf{x}) = \sum_{n=1}^{N_{\text{mod}}} \sum_{m=-n}^{n} p_{n,m}^{j,\perp} \nabla_{\mathbb{S}^2} Y_{n,m}(\widehat{x}_j) + p_{n,m}^{j,\times} \mathbf{curl}_{\mathbb{S}^2} Y_{n,m}(\widehat{x}_j) \qquad \mathbf{x} \in \Gamma_\delta^j$$

with $\widehat{x}_j = \frac{\mathbf{x} - c_j}{|\mathbf{x} - c_j|}$ and $\nabla_{\mathbb{S}^2} Y_{n,m}$, $\mathbf{curl}_{\mathbb{S}^2} Y_{n,m}$: complex-valued vector spherical harmonics

Vectorial formulation

- Variational formulation: Find $\mathbf{p}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$ such that

$$\sum_{j=1}^{N_{\mathrm{obs}}} \langle \mathcal{M}_\Gamma^{kj} \mathbf{p}_j, \mathbf{v}_k \rangle_{\Gamma_\delta^k} = -\langle \mathbf{n} \times \mathbf{E}^{\mathrm{inc}}, \mathbf{v}_k \rangle_{\Gamma_\delta^k} \qquad \forall\, \mathbf{v}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\mathrm{curl}_{\Gamma_\delta^k}, \Gamma_\delta^k)$$

- Vectorial formulation: Find $\mathbf{p} = \left((p_{n,m}^{1,\perp}), \dots, (p_{n,m}^{N_{\mathrm{obs}},\perp}), (p_{n,m}^{1,\times}), \dots, (p_{n,m}^{N_{\mathrm{obs}},\times})\right)^\top \in \mathbb{C}^{\boldsymbol{N}}$ s.t.

$$\mathbb{M}\, \mathbf{p} = \mathbf{f}$$

with $\boldsymbol{N} = 2N_{\mathrm{mod}}(N_{\mathrm{mod}} + 2)N_{\mathrm{obs}}$ and

$$\mathbb{M} = \begin{pmatrix} \mathbb{M}_{\perp\perp} & \mathbb{M}_{\perp\times} \\ \mathbb{M}_{\times\perp} & \mathbb{M}_{\times\times} \end{pmatrix} \qquad \text{with} \qquad \mathbb{M}_{\alpha\beta} = \begin{pmatrix} \mathbb{M}_{\alpha\beta}^{11} & \mathbb{M}_{\alpha\beta}^{12} & \dots & \mathbb{M}_{\alpha\beta}^{1N_{\mathrm{obs}}} \\ \mathbb{M}_{\alpha\beta}^{21} & \mathbb{M}_{\alpha\beta}^{22} & \dots & \mathbb{M}_{\alpha\beta}^{2N_{\mathrm{obs}}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{M}_{\alpha\beta}^{N_{\mathrm{obs}}1} & \mathbb{M}_{\alpha\beta}^{N_{\mathrm{obs}}2} & \dots & \mathbb{M}_{\alpha\beta}^{N_{\mathrm{obs}}N_{\mathrm{obs}}} \end{pmatrix}$$

$$= \mathbb{M}_{\alpha\beta}(\delta, \mathbf{d}_{jk})$$

Vectorial formulation

- Variational formulation: Find $\mathbf{p}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$ such that

$$\sum_{j=1}^{N_{\mathrm{obs}}} \langle \mathcal{M}_\Gamma^{kj} \mathbf{p}_j, \mathbf{v}_k \rangle_{\Gamma_\delta^k} = -\langle \mathbf{n} \times \mathbf{E}^{\mathrm{inc}}, \mathbf{v}_k \rangle_{\Gamma_\delta^k} \qquad \forall\, \mathbf{v}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\mathrm{curl}_{\Gamma_\delta^k}, \Gamma_\delta^k)$$

- Vectorial formulation: Find $\mathbf{p} = \left((p_{n,m}^{1,\perp}), \ldots, (p_{n,m}^{N_{\mathrm{obs}},\perp}), (p_{n,m}^{1,\times}), \ldots, (p_{n,m}^{N_{\mathrm{obs}},\times})\right)^\top \in \mathbb{C}^{\boldsymbol{N}}$ s.t.

$$\mathbb{M}\,\mathbf{p} = \mathbf{f}$$

Computation time
Ill-conditionned system
Memory consumption

- Numerical integration (Gauss-Lobatto)
- Large number of unknowns
- Dense matrix

Mex interface (C++)
Preconditionning (system & matrix)
Smart storage and assembling

- Code speed-up
- Linear algebra tools
- Sub-blocks $\mathbb{M}_{\alpha\beta}^{kj}$ depend on $\mathbf{d}_{jk}$

Preconditionning

Linear system: Change of variable + Permutation + Normalization

$\mathbf{E}_\delta = \phi_\delta(\mathbf{p}_\delta)$ with $\mathbb{M}\,\mathbf{p}_\delta = \mathbf{f}$ $\longrightarrow$ $\mathbf{E}_\delta = \phi(\widetilde{\mathbf{p}}_\delta)$ with $\widetilde{\mathbb{M}}\,\widetilde{\mathbf{p}}_\delta = \widetilde{\mathbf{f}}$

Dense matrix $\widetilde{\mathbb{M}}$

Error of approximation
Analytic preconditionner (*e.g.* dipole)
Algebraic preconditionner (small criterion)

Preconditionning

Linear system: Change of variable + Permutation + Normalization

$\mathbf{E}_\delta = \phi_\delta(\mathbf{p}_\delta)$ with $\mathbb{M}\,\mathbf{p}_\delta = \mathbf{f}$ $\longrightarrow$ $\mathbf{E}_\delta = \phi(\widetilde{\mathbf{p}}_\delta)$ with $\widetilde{\mathbb{M}}\,\widetilde{\mathbf{p}}_\delta = \widetilde{\mathbf{f}}$

Dense matrix $\widetilde{\mathbb{M}}$

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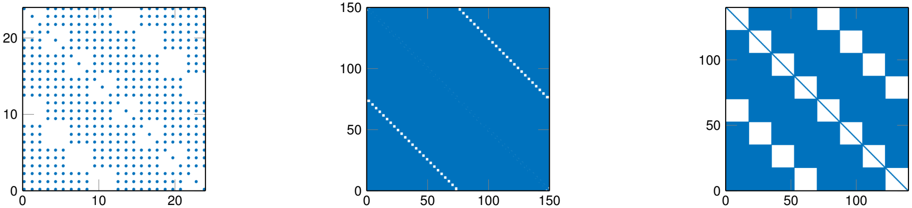
Analytic preconditionner (*e.g.* dipole)
Algebraic preconditionner (small criterion)



(a) $N_{\rm obs} = 4$ and $N_{\rm mod} = 1$ (b) $N_{\rm obs} = 25$ and $N_{\rm mod} = 1$ (c) $N_{\rm obs} = 4$ and $N_{\rm mod} = 5$

Figure: Skeletton of $\widetilde{\mathbb{M}}$

Preconditionning

Linear system: Change of variable + Permutation + Normalization

$\mathbf{E}_\delta = \phi_\delta(\mathbf{p}_\delta)$ with $\mathbb{M}\,\mathbf{p}_\delta = \mathbf{f}$ $\longrightarrow$ $\mathbf{E}_\delta = \phi(\widetilde{\mathbf{p}}_\delta)$ with $\widetilde{\mathbb{M}}\,\widetilde{\mathbf{p}}_\delta = \widetilde{\mathbf{f}}$

Dense matrix $\widetilde{\mathbb{M}}$

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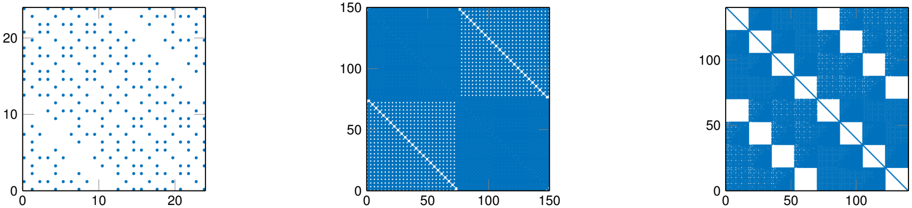


(a) $N_{\text{obs}} = 4$ and $N_{\text{mod}} = 1$ (b) $N_{\text{obs}} = 25$ and $N_{\text{mod}} = 1$ (c) $N_{\text{obs}} = 4$ and $N_{\text{mod}} = 5$

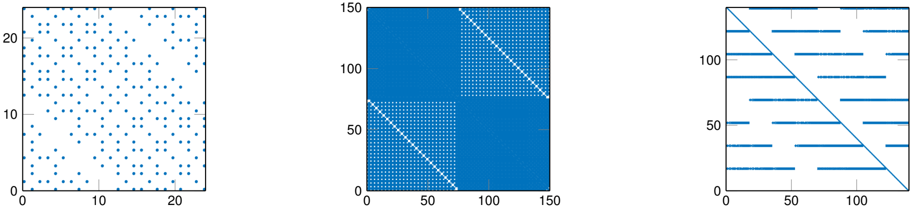
Figure: Skeletton of $\widetilde{\mathbb{M}}$

Preconditionning

Linear system: Change of variable + Permutation + Normalization

$\mathbf{E}_\delta = \phi_\delta(\mathbf{p}_\delta)$ with $\mathbb{M}\,\mathbf{p}_\delta = \mathbf{f}$ $\longrightarrow$ $\mathbf{E}_\delta = \phi(\widetilde{\mathbf{p}}_\delta)$ with $\widetilde{\mathbb{M}}\,\widetilde{\mathbf{p}}_\delta = \widetilde{\mathbf{f}}$

- Dense matrix $\widetilde{\mathbb{M}}$
- Error of approximation
  Analytic preconditionner (*e.g.* dipole)
  Algebraic preconditionner (small criterion)



(a) $N_{\text{obs}} = 4$ and $N_{\text{mod}} = 1$ (b) $N_{\text{obs}} = 25$ and $N_{\text{mod}} = 1$ (c) $N_{\text{obs}} = 4$ and $N_{\text{mod}} = 5$

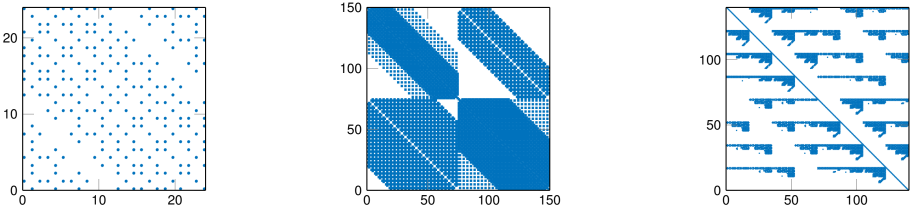
Figure: Skeletton of Precond($\widetilde{\mathbb{M}}$)

Preconditionning

Linear system: Change of variable + Permutation + Normalization

$$\mathbf{E}_\delta = \phi_\delta(\mathbf{p}_\delta) \quad \text{with } \mathbb{M}\,\mathbf{p}_\delta = \mathbf{f} \longrightarrow \mathbf{E}_\delta = \phi(\widetilde{\mathbf{p}}_\delta) \quad \text{with } \widetilde{\mathbb{M}}\,\widetilde{\mathbf{p}}_\delta = \widetilde{\mathbf{f}}$$

- Dense matrix $\widetilde{\mathbb{M}}$
- Error of approximation
  Analytic preconditionner (*e.g.* dipole)
  Algebraic preconditionner (small criterion)



(a) $N_{\text{obs}} = 4$ and $N_{\text{mod}} = 1$ (b) $N_{\text{obs}} = 25$ and $N_{\text{mod}} = 1$ (c) $N_{\text{obs}} = 4$ and $N_{\text{mod}} = 5$

Figure: Skeletton of Precond($\widetilde{\mathbb{M}}$)

Smart storage and assembling

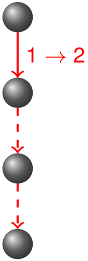
- Example: 4 aligned obstacles

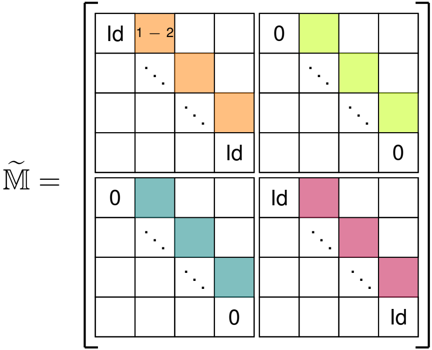
$$\widetilde{\mathbb{M}} = \left[\begin{array}{cccc|cccc} \mathrm{Id} & & & & 0 & & & \\ & \ddots & & & & \ddots & & \\ & & \ddots & & & & \ddots & \\ & & & \mathrm{Id} & & & & 0 \\ \hline 0 & & & & \mathrm{Id} & & & \\ & \ddots & & & & \ddots & & \\ & & \ddots & & & & \ddots & \\ & & & 0 & & & & \mathrm{Id} \end{array}\right]$$

The sub-blocks $\mathbb{M}^{kj}_{\alpha\beta}$ depend only on δ and $\mathbf{d}_{jk}$

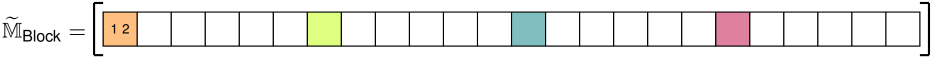
Smart storage and assembling

- Example: 4 aligned obstacles





- Storage of sub-blocks



Smart storage and assembling

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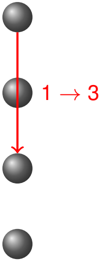
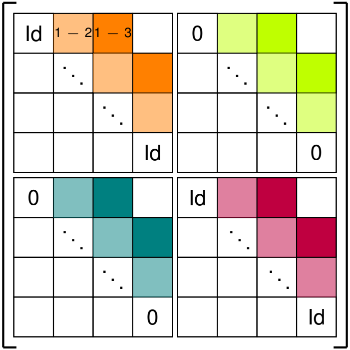
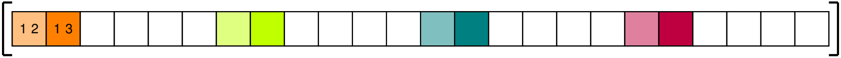


$\widetilde{\mathbb{M}} =$



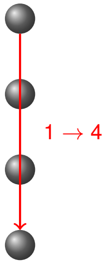
- Storage of sub-blocks

$\widetilde{\mathbb{M}}_{\text{Block}} =$

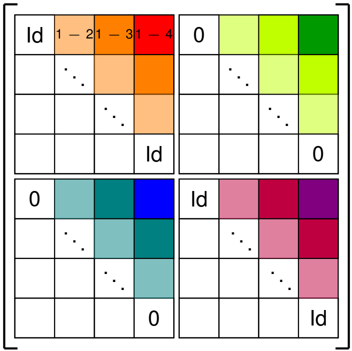


Smart storage and assembling

- Example: 4 aligned obstacles

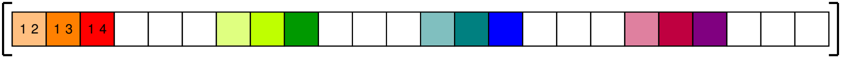


$\widetilde{\mathbb{M}} =$



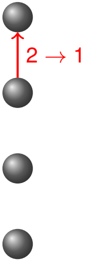
- Storage of sub-blocks

$\widetilde{\mathbb{M}}_{\text{Block}} =$

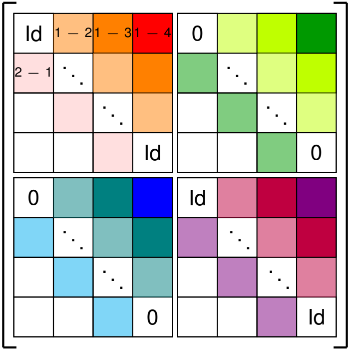


Smart storage and assembling

- Example: 4 aligned obstacles

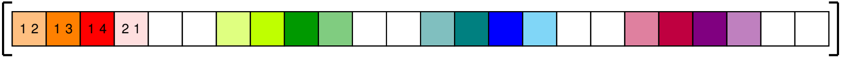


$\widetilde{\mathbb{M}} =$



- Storage of sub-blocks

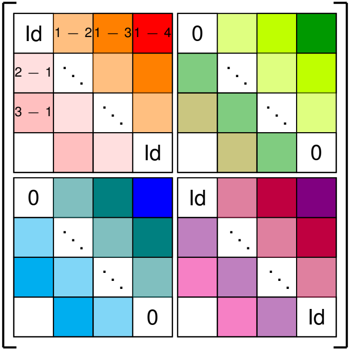
$\widetilde{\mathbb{M}}_{\text{Block}} =$



Smart storage and assembling

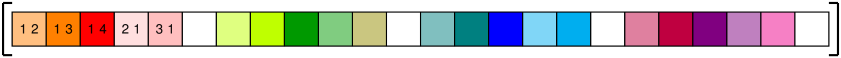
- Example: 4 aligned obstacles

$\widetilde{\mathbb{M}} =$



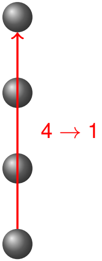
- Storage of sub-blocks

$\widetilde{\mathbb{M}}_{\text{Block}} =$

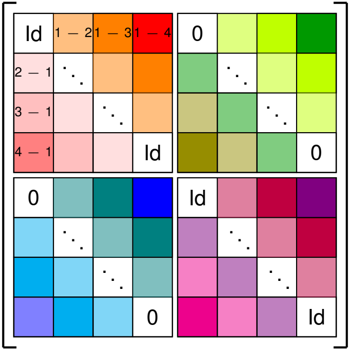


Smart storage and assembling

- Example: 4 aligned obstacles



$\widetilde{\mathbb{M}} =$



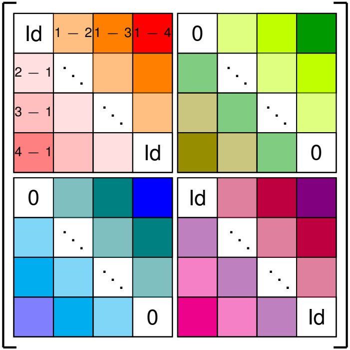
- Storage of sub-blocks

$\widetilde{\mathbb{M}}_{\text{Block}} =$

Smart storage and assembling

- Example: 4 aligned obstacles

$\widetilde{\mathbb{M}} =$



- Storage of sub-blocks

$\widetilde{\mathbb{M}}_{\text{Block}} =$

- Iterative solvers: define the action of $\widetilde{\mathbb{M}}_{\text{Block}}$ on $\mathbf{u}$

$$\mathcal{A}(\widetilde{\mathbb{M}}_{\text{Block}}, \mathbf{u}) := \widetilde{\mathbb{M}}\,\mathbf{u}$$

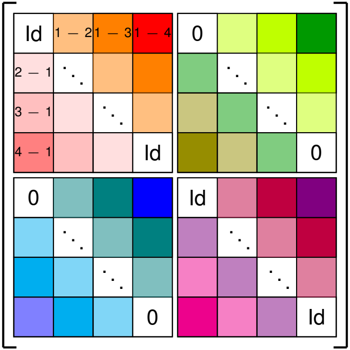
Smart storage and assembling

- Example: 4 aligned obstacles

$8(N_{\text{obs}} - 1)$ blocks

instead of $4(N_{\text{obs}})^2$

$\widetilde{\mathbb{M}} =$



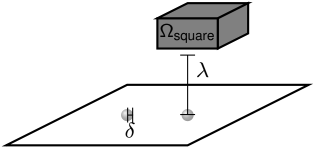
- Storage of sub-blocks

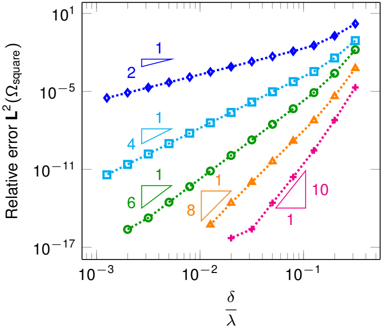
$\widetilde{\mathbb{M}}_{\text{Block}} =$

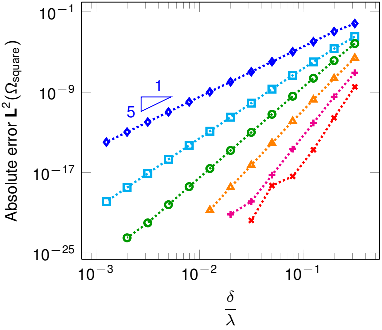
N_{obs}	$4(N_{\text{obs}})^2$	$8(N_{\text{obs}} - 1)$	Ratio (%)	$\widetilde{\mathbb{M}}$ (GB)	$\widetilde{\mathbb{M}}_{\text{Block}}$ (GB)
100	40 000	792	1.98	$5.76 \cdot 10^{-3}$	$1.15 \cdot 10^{-4}$
1 000	4 000 000	7 992	$1.998 \cdot 10^{-3}$	$5.76 \cdot 10^{-1}$	$1.15 \cdot 10^{-3}$
3 000	36 000 000	23 992	$6.664 \cdot 10^{-4}$	4.01	$3.45 \cdot 10^{-3}$
10 000	400 000 000	79 992	$1.9998 \cdot 10^{-4}$	- -	$1.15 \cdot 10^{-2}$

Numerical validation of spectral solutions

- Incident field: Electromagnetic plane wave along z-axis polarized by $\boldsymbol{e}_x$
- Reference solution: Spectral solution with $N_{\text{mod}} = 10$

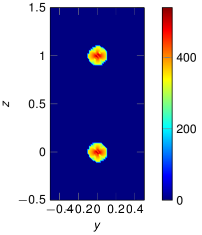




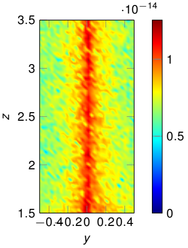


Comparison with asymptotic models

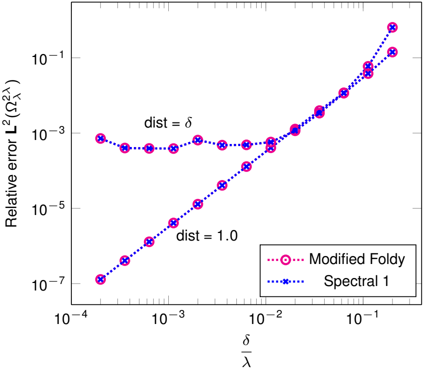
Difference between Spectral 1 and modified Foldy



(c) Close to the obstacles



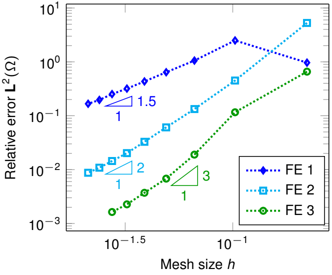
(d) In the far-field domain

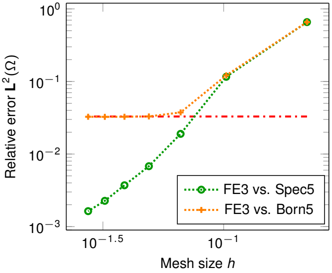


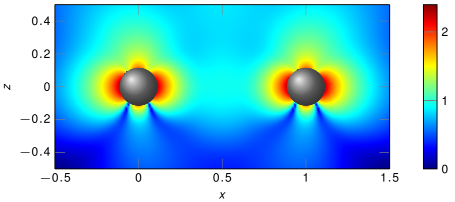
	Modified Foldy				Spectral 1			
N_{obs}	100	1000	5000	10000	100	1000	5000	10000
Linear system	0.17	17.71	34.01	43.49	2.66	40.74	352.47	- -
Post-processing	5.25	47.84	254.34	459.90	12.17	126.81	$>$ 3600	- -

Table: Time comparison in seconds

Comparison with finite element solutions (Montjoie)







	Spectral 3		
N_{obs}	1	2	4
Size of linear system	30	60	120
$\mathbf{L}^2$-Accuracy ($\times 10^{-6}$)	4.77	4.48	3.66
Time (s)	15.46	65.55	262.33
Memory (GB)	0.3	2.36	9.3

	Finite Element 3		
N_{obs}	1	2	4
Size of linear system	925 422	1 972 800	3 942 936
$\mathbf{L}^2$-Accuracy ($\times 10^{-3}$)	0.28	2.27	3.31
Time (min)	26.5	54.25	147.25
Memory (GB)	21.11	36.97	75

Table: Computational costs

Outline

Conclusions and Perspectives

Conclusion

- ✓ Derivation and numerical validation of the matched asymptotic expansions
- ✓ Extension to Born and Foldy-Lax models
- ✓ Derivation and implementation at any order spectral method + Numerical validation

On-going work

- ∼ Comparison of preconditionners and iterative solvers
- ∼ Smart assembling for particular configurations (plane, cubic volume)
- ∼ Contribution to mathematical justification of the asymptotic expansions

Perspectives

- ✗ Existence of electromagnetic centers?
- ✗ Extension to obstacles of arbitrary shape
- ✗ Extension to time-dependent domain

Thank you for your attention